EXPERIMENT 14
Variable-frequency networks

The objective of this experiment is to:
- Investigate networks excited with variable-frequency sinusoidal signals

I. Introduction

The ac steady-state behavior of circuits depends on the frequency of the sinusoidal input. The impedances of all of the elements in the circuit, except for resistors, are dependent on the frequency of the excitation. The impedance of a capacitor is

\[ Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C} = jX_C, \quad X_C \equiv -\frac{1}{\omega C}. \]

\( \frac{1}{\omega C} \) is infinite for \( \omega = 0 \) (zero frequency or dc) and reduces to a small value for very large values of frequency. Therefore, the capacitor behaves in the steady-state analysis of a circuit as an open-circuit to dc and as a short-circuit for very large frequencies. The impedance of an inductor is

\[ Z_L = j\omega L = jX_L, \quad X_L = \omega L. \]

\( \omega L \) is zero for zero frequency and approaches infinity as the frequency becomes larger and larger. Therefore, the inductor behaves in the steady-state analysis of a circuit as a short-circuit to dc and becomes an open-circuit for very high frequencies.

Combinations of inductors, capacitors and resistors connected together to form a circuit can produce interesting behavior as a function of frequency. As the frequency of the excitation to a circuit increases, the resistance of the resistors stays the same, the \( \omega L \)'s of the inductors increase while the \( 1/(\omega C)'s \) of the capacitors decrease. Three such circuits that are classified as simple frequency filters will be experimentally examined in this exercise. Each filter will be tested by placing a fixed amplitude sinusoidal voltage on the input to the filter and measuring the output of the filter as the frequency of the input sinusoidal voltage is varied. The phasor output voltage of the filter, divided by the phasor input voltage, is defined as the voltage transfer function of the filter. Since this voltage ratio will depend on the frequency of the input voltage, the transfer function is a
function of frequency, i.e., the transfer function is defined as \( G(f) \equiv \frac{V_{\text{out}}}{V_{\text{in}}} \) which is a function of frequency. The magnitude of the transfer function, i.e., the magnitude of the output voltage divided by the magnitude of the input voltage, will be plotted for each filter as a function of the frequency of the input sinusoidal voltage.

The analysis of a frequency selective circuit is nothing more than a study of the ac steady-state solution of the circuit at several values of frequency. Most of the time this analysis is performed with the frequency \( f \), or \( \omega = 2\pi f \), treated as a variable. A solution is obtained in the form of an expression \( G(f) \), or \( G(\omega) \), that is a function of frequency. Then this expression is evaluated for several values of frequency to determine how the circuit behaves.

The first circuit, shown in Figure 1, is a low pass filter. To illustrate how frequency-domain analysis is used to analyze the low pass filter, the transfer function is derived under the condition that \( R_L << R \).

\[
V_{\text{out}} = \frac{V_{\text{in}}(R)}{R_L + R + j\omega L} \approx \frac{V_{\text{in}}(R)}{R + j\omega L}
\]

\[
G(\omega) \equiv \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R}{R + j\omega L} \quad \text{or} \quad G(f) = \frac{R}{R + j(2\pi f) L}
\]

Note that for a dc input, \( \omega = 0 \), the inductor is a short and the output voltage is the same as the input voltage making the transfer function \( G(0) = 1 \). As the frequency increases, \( \omega L \) increases, the denominator increases and the output voltage decreases. When the frequency is high enough to consider the inductor as an open circuit, \( \omega L >> R \), the output voltage approaches zero, therefore \( G(\omega) \to 0 \). The circuit is called a low-pass filter, because the output voltage is equal to the input voltage for low frequencies and the output voltage is reduced from the input voltage as the frequency is increased. A low-pass filter will pass low-frequency signals and attenuate high frequency signals.

The second circuit, shown in Figure 2, has a voltage transfer function that is one for high frequencies and approaches zero as the frequency is decreased, reaching zero when the frequency is zero. The circuit is called a high-pass filter because the output voltage is equal to the input voltage for high frequencies and the output voltage is reduced below the input voltage for low frequencies. A high-pass filter will attenuate low frequency signals and pass high frequency signals.
The third circuit, shown in Figure 3, is a band-pass filter. An analysis of this circuit shows that: 1) at low frequencies the impedance of the capacitor is large; 2) at high frequencies the impedance of the inductor is large, and 3) at some mid frequency \( \omega_L \) and \(-1/\omega_C\) will cancel making the output voltage equal to the input voltage. Therefore, the filter will attenuate low- and high-frequency signals and pass the signals in a band of frequencies in a middle range. This mid range band of frequencies is called the pass-band of the band-pass filter.

Filters are used to select signals that are in a desired frequency range and reject those that are not desired. For example, a radio receiver uses a band-pass filter to separate the desired radio station from all of the others stations. To change stations the pass-band is moved to the frequency range of the new station. Telephones use a filter that will reject 60 Hz signals that are pick up noise from the power system. Without this filter the telephone would have a 60 Hz buzz at all times.

It is often convenient to define standard quantities that will simplify the analysis and improve the understanding of filter circuits. A particular frequency that is often defined for filters is the value of frequency that will make the magnitude of the voltage transfer function equal to .707, i.e., \(|G(f_c)| = 0.707\). This frequency is
called the **cutoff frequency**. A reduction from 1 to 0.707 in the magnitude of $G(f)$ is a reduction of 3db ($20\log_{10} 1 - 20\log_{10} 0.707 = 3$). Therefore, the cutoff frequency is also called the **3db frequency**. When the voltage is reduced from 1 to 0.707, the power, which is the square of the voltage, is reduced from $P$ to $(1/2)P$. Therefore, the cutoff frequency is also called the **half-power frequency**. Regardless of the name, it is the labeled $f_c$ and is the frequency where

$$
|G(f_c)| = \frac{1}{\sqrt{2}} = \left| \frac{1}{1 \pm j} \right| = \left| \frac{1}{1 + jX} \right|_{x = \pm 1}
$$

The transfer function of the low-pass filter in Figure 1 with $R_L << R$ can be written as

$$
G(f) = \frac{R}{R + j(2\pi f)L} = \frac{1}{1 + j \frac{(2\pi f)L}{R}} = \frac{1}{1 + j \frac{f}{f_c}}
$$

where the cutoff or half-power frequency $f_c = R/(2\pi L)$.

The transfer function of the high-pass filter shown in Figure 2 can be written as

$$
G(f) = \frac{R}{R + \frac{i}{j\omega C}} = \frac{1}{1 - j \frac{1}{2\pi fC}} = \frac{1}{1 - j \frac{f}{f_c}}
$$

where the cutoff or half-power frequency $f_c = 1/(2\pi RC)$.

The transfer function of the band-pass filter in Figure 3 with $R_L << R$ can be written as

$$
G(\omega) = \frac{R}{R + \frac{1}{j\omega C} + j\omega L} = \frac{1}{1 + j\left(\omega L - \frac{1}{\omega C}\right)}
$$

The band-pass filter has two half-power frequencies, one at the low end of the pass-band and one at the high end. These half-power frequencies are:

$$
f_{c1} = \frac{1}{4\pi L} \left[ -R + \sqrt{R^2 + \frac{4L}{C}} \right] \text{ for the low frequency}
$$

$$
f_{c2} = \frac{1}{4\pi L} \left[ +R + \sqrt{R^2 + \frac{4L}{C}} \right] \text{ for the high frequency.}
$$

Note that in Equation (3) the inductive reactance $X_L$ will cancel with the
capacitive reactance $X_C$ when $\omega_r L = 1/(\omega_r C)$ giving $G(\omega_r) = 1$. When this occurs the input voltage and current are in phase and the circuit is said to be in resonance. The frequency at which this occurs is called the resonant frequency and is

$$\omega_r = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_r = \frac{1}{2\pi\sqrt{LC}}$$

(5)

Another quantity that is convenient to define for the series-resonant circuit is the $Q$ of the circuit. The $Q$ is a measure of how narrow the pass-band is and is defined as

$$Q = \frac{\text{resonant frequency}}{\text{width of pass-band}} = \frac{f_r}{f_{c2} - f_{c1}}$$

(6)

The $Q$ can also be shown to be $Q = \frac{\omega_r L}{R}$. The voltage across the capacitor when the input frequency is the resonant frequency can be related to $Q$ by

$$Q = \frac{|V_C|}{|V_{in}|}_{f=f_r} = \frac{\text{magnitude of } V_C \text{ at } f = f_r}{\text{magnitude of input voltage}}$$

(7)

For example if a series RLC circuit with a $Q$ of 200 and a resonant frequency of 1,000 Hz has an input sinusoidal voltage with a magnitude of 10 volts at a frequency of 1,000 Hz, then the voltage across the capacitor will be 2,000 volts. 10 volts in, 2,000 volts out? Yes! Be careful when dealing with high Q circuits!

II. Lab Exercises

The digital multimeter will be utilized as a voltmeter for all voltage measurements in this experiment. The test setup is shown in Figure 4.

![Figure 4 Test circuit for measuring the voltage transfer functions of the filters tested in this experiment.](image)

1) Construct the test circuit in Figure 4 where the filter is the low-pass filter shown in Figure 1 with $R = 1,000 \ \Omega$ and $L=0.4 \ \text{H}$. Use the sine wave output of the function generator as the input voltage source. The digital voltmeter is connected
to the circuit using a single pole, two position switch so that the meter reads the magnitude of the input voltage in position "A" and the magnitude of the output voltage when in position "B".

Measure the magnitude of the voltage transfer function of the low-pass filter for frequencies from 100 to 10,000 Hz. Use an input voltage close to the maximum output of the signal generator, and it will be necessary to measure the input voltage at each frequency. Measure enough points to plot a smooth curve but not an excess number of points. Make sure that one of your measured points is approximately at the frequency $f_c$ so that the magnitude of the transfer function is approximately 0.707. Record the frequency, input voltage magnitude, and output voltage magnitude in a table. Calculate the magnitude of the voltage transfer function and record in the same table. Enter the measured frequency data in a vector called $f$ and the magnitude data in a vector called $g$. Plot the magnitude of the voltage transfer function using MATLAB using the following MATLAB commands:

```matlab
>> semilogx(f,g,'+-');
>> grid;
>> xlabel('f(Hz)');
>> ylabel('|G(f)|');
>> title('Transfer Function Magnitude');
```

Include this plot in your report. Scale the magnitude data into decibels (dB) as plot again using the commands shown below:

```matlab
>> gdb=20*log10(g);
>> semilogx(f,gdb,'+-');
>> grid;
>> xlabel('f(Hz)');
>> ylabel('|G(f)| (dB)');
>> title('Transfer Function Magnitude');
```

Include this plot in your report.

2) Change the $L$ to $C = 0.1 \ \mu F$ to give the high-pass filter shown in Figure 2 and repeat 1).

3) Construct the band-pass filter of Figure 3 using $C = 0.1 \ \mu F$, $L = 0.4 \ \text{H}$, and $R =$
1,000 \, \Omega \text{ and repeat 1). Make sure that two of your measured points are approximately at the frequencies } f_{c1} \text{ and } f_{c2} \text{ where the magnitude of the transfer function is approximately 0.707.}